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The credit risk in SME loans portfolios: Modeling issues, pricing, and capital requirements

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Abstract

This paper is devoted to the credit risk modeling issues of small commercial loans portfolios. We propose specific solutions dealing with the most important peculiarities of these portfolios: their large size and the limited information about the financial situation of borrowers. We then compute the probability density function of futures losses and VaR measures in a portfolio of 220.000 French SMEs. We also compute marginal risk contributions in order to discuss the loan pricing issue of small commercial loans and to compare the capital requirements derived from our model with those derived from the New Ratings-Based Basel Capital Accord. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

During recent years, financial institutions have devoted important resources to build statistical models to measure the potential losses in their loans

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portfolios. Supervisors have recognized such efforts. The New Basel Capital Accord allows banks to compute the minimum capital requirements using an internal ratings based (IRB) approach which is founded on the most sophisticated credit risk internal models. However, most of the current models have considered the credit risk in wholesale commercial loans portfolios. Few attempts have been devoted to small commercial loans credit risk, despite the relatively high share of SME exposures in the banks loans portfolios, especially in Europe. The first objective of this paper is to present a Value at Risk model of the SME credit risk dealing with the specific methodological problems which arise in the modeling of small commercial loans portfolios.

Several factors distinguish credit risk in small and wholesale commercial loans portfolios. First, the primary credit risk of small business loans is that they will not be repaid. The SME credits are not traded in organized financial markets and their value does not change until maturity, except if the borrower defaults. This restricts the modeling choice to the default mode paradigm (Jones, 1998), while wholesale commercial loans credit risk models are "multi-state" or "Marked-To-Market" models which incorporate transition probabilities between non-default rating classes. Second, the size of a small commercial loans portfolio is larger than that of a large corporate loans portfolio. While the latter contains hundreds of loans, the former contains thousands of loans. Consequently, it is very difficult to adopt the methodological choices which are used in the models dealing with large corporate exposures, like CreditMetrics, CreditRisk+ or KMV. In particular, it would consume too much time to simulate directly potential losses at the individual level, as it is the case in the CreditMetrics model (Bathia et al., 1997), for example. Methodological choices are restricted by time constraints. Consequently, a retail credit risk model should proceed in two separate steps: the first step should be devoted to the simulation of the number of defaults into each risk class, the second step to the simulation of the amount of individual losses given default (LGD). Third, data limitations also restrict the modeling choices. The wholesale commercial loans models use a rich information concerning companies financial health which comes from rating agencies and financial markets prices. In general, this information is available in the form of time series. It allows to assess the long run stability of the main building blocks of any credit risk model (default probabilities (PDs), loans losses given default, and correlations). It also allows to derive analytically the probability distribution of potential losses or to proceed to historical simulations. In the small business case, the relevant information is reduced to default scores. This is the reason why the credit risk model should build the loss density function by using the information given by the volatility of scores.

The final objective of any credit risk model is to build the probability density function (PDF) of future losses in a loans portfolio. In the case of small commercial loans portfolios, this implies to find specific solutions to the two main problems of the modeling process: the modeling of the PD, and the construction of the joint distribution of losses taking into account correlations between default in the portfolio. Concerning the first problem, we present two alternative frameworks to model PD, a probit model and a model using a gamma distribution. Concerning the second problem, we propose specific solutions to compute the PDF in large commercial loans portfolios. We applied these solutions by using a panel of more than 220.000 French SME provided by a large French credit insurance company (Coface SCRL). For each small business, this panel gives two types of information. First information is a default score. This allows to rank borrowers in risk classes and to compute the PDs. The second is the balance sheet amount of the firm bank debt. This panel contains more than two thirds of all incorporated French SME. Consequently, it is as if we had measured the credit risk exposure of a large single bank which would own nearly all the French small commercial loans.

The paper is organized as follows: Section 2 presents the general structure of a small commercial loans credit risk model and the methodological solutions dealing with the peculiarities of small commercial loans portfolios. Section 3 presents measures of the PDF of losses in a large portfolio of French SME loans, and it derives VaR measures for credit risk. Section 4 is devoted to the calculation of marginal risk contributions and to the issue of economic capital allocation. Section 5 compares the bank capital requirements derived from our internal SME model with those derived from the new IRB "advanced" approach proposed by the Basel Committee (Basel Committee on Banking Supervision, 2001). Section 6 discusses the loan pricing issue of SME commercial loans and Section 7 concludes the paper.

2. The SME credit risk model

There are two steps in the construction of the PDF. In the first step, we should characterize each individual exposure in the bank's credit portfolio. That means, first, to classify the individual loans by their credit quality. The credit quality grade is obtained by distributing the loans across the bank internal risk rating system, which assigns each loan to one of the risk classes. In a default mode model, this is used to assess the probability that the borrower will move to the default class over a given planning horizon. So, each borrower is characterized by its long run PD. [2] Second, to measure the likely exposure of each loan in a default mode model, we simply assume that the exposure is the

[2] Formally, that means that the stationary probability, $\bar{p}$ in the rest of the text, is a weighted mean of the default frequencies observed over several time periods in a given class of borrowers. Moreover, this weighted mean is the ML estimator of the stationary default probability (see, for example, Maddala, 1983).

face value of the commercial loan. [3] Finally, the last input is the measurement of the LGD, that is the proportion of the exposure that would be lost if the borrower defaults. That mainly implies to compute the recovery rate for each loan. Here, we do not consider this issue explicitly.

The second step in generating the PDF of future losses requires to bring the individual loans together. To capture the behavior of the portfolio as a whole, it is necessary to measure the correlations between losses. The more the individual loans tend to vary jointly, the higher the portfolio credit risk will be. Like most credit risk models, we assume that exposures and default are independent and we then focus on the correlations between defaults. In a default mode model, the integration of correlations is made by measuring variations of PD in each risk class around its long run stationary value. Moreover, most of the credit risk models assume that the variances of the PDs and the correlations are driven by one or several "risk factors" that represent various sources of change in the borrower's financial situation (business cycle, sector, geographic location, for example). Here, it is important to note that the assumptions about the distribution of the risk factors determine the general structure of the model and the shape of the PDF (Gordy, 2000a). So, in what follows, we will use two different frameworks in order to evaluate the actual consequences of alternative assumptions on the amount of potential losses in small commercial loans portfolios.

In generating the PDF of a small commercial loans portfolio, constraints coming from the large size of the portfolio impose to proceed in two separate steps. The first step is devoted to the modeling of the PD and the simulation of the number of defaults, the second step to the simulation of the joint individual losses.

2.1. Two alternative models

Here, the objective is to define the distribution of the PD in each risk class. In order to test the robustness of the results when using various statistical processes governing the risk factors, we build two alternative models: the ordered probit model and a model which assumes a gamma distribution of the systematic factor. These two models are also close to the current wholesale commercial loans models of credit risk. The first framework is used by the CreditMetrics model. The second model is close to the CreditRisk+ model. So, by using these two different frameworks, we also verify the capacity of the most usual frameworks to measure potential losses in retail as well as in wholesale

[3] The credit exposure can be defined as the maximum potential loss on a credit instrument in case of default. So, defining exposure can be a challenging task when the credit instruments composing the portfolio entail embedded options or other instruments with non-linear face value.

portfolios. In what follows, we assume that each exposure is measured by the facial value of the loan. We also assume that the LGD is exogenous.

2.1.1. The ordered probit model

In this model, each borrower's financial position at the end of a planning horizon (default or not) is determined by one systematic risk factor and one idiosyncratic risk factor. Indeed, the end of period borrower i state is driven by an unobserved latent random variable U, which is defined as a linear function of a single systematic factor x and a specific idiosyncratic factor ε_i:

$$U = wx + \varepsilon_i \tag{1}$$

where x and ε_i are standard normal variables and $E[x\varepsilon_i] = 0$. The systematic factor represents the state of the economy. It measures the effect of the business cycle on the default rate. The state of the borrower at the end of the planning horizon depends on the location of the latent variable relative to a "cut-off" value, which defines default. If the latent variable is a standard normal variable, the default cut-off value is set so that the unconditional PD for a borrower belonging to a given risk class is $\bar{p}$. The cut-off value is simply $\Phi^{-1}(\bar{p})$, where $\Phi(\cdot)$ is the standard normal CDF. Let us define Z_i as the standardized latent variable of borrower i. Therefore, a borrower makes default when

$$\frac{wx + \varepsilon_i}{\sqrt{1 + w^2}} < \Phi^{-1}(\bar{p}), \tag{2}$$

or, alternatively, for a given value of the systematic factor x,

$$\varepsilon_i < \sqrt{1 + w^2}\Phi^{-1}(\bar{p}) - wx. \tag{3}$$

This condition allows to compute $p(x)$, the individual PD conditional to the realization of the systematic factor x. This probability is simply derived from Eq. (3), as follows:

$$\begin{aligned} p(x) &= \Pr\left\lfloor \varepsilon_i < \sqrt{1 + w^2}\Phi^{-1}(\bar{p}) - wx \right\rfloor \\ &= \Phi\left\lfloor \sqrt{1 + w^2}\Phi^{-1}(\bar{p}) - wx \right\rfloor \text{ with } \varepsilon_i \sim N(0;1). \end{aligned} \tag{4}$$

This is the conditional PD of a borrower with stationary probability $\bar{p}$. If the realization of the systematic factor is good (that is, if the economy is going well), the firm will default only if the realization of the specific idiosyncratic factor is worse. Otherwise, the (standardized) latent variable Z_i will not cross the default cut-off value $\Phi^{-1}(\bar{p})$. The value of $p(x)$ fluctuates around the stationary probability depending of the values of the systematic risk factor and of w. Moreover, the degree of correlation between defaults is determined by the sensibility of the latent variables to the systematic factor, that is by w. For two borrowers i and j with the same rating grade, the (non-conditional) covariance between latent variables is given by

$$\mathrm{Cov}[Z_i; Z_j] = E[Z_i Z_j] - E[Z_i]E[Z_j] = \frac{w^2}{1+w^2}. \tag{5}$$

Therefore, correlation between latent variables is due to the existence of aggregate shocks in the economy. In addition, in the probit model, it is the existence of correlation between defaults that determines (at least partially) the shape of the end of period value distribution of the portfolio. We have verified that the shape of this distribution is skewed to the right, and that its degree of asymmetry directly depends on the weight w of the systematic risk factor.

2.1.2. The gamma distribution model

The probit default model assumes that the distribution of the risk factors is normal. An alternative assumption is that the risk factors are gamma distributed with unit mean and variance σ^2 (this is also the case in the CreditRisk+ model). As before, the default rate is driven by a systematic and a specific risk factor. Using previous notations, the conditional PD takes the following multiplicative form:

$$p(x) = \bar{p}(wx + (1-w)\varepsilon_i). \tag{6}$$

If we assume that the specific risk can be diversified away, that means that it does not contribute to the overall portfolio variance. Hence, the specific factor is constant and equal to unity. The conditional PD becomes

$$p(x) = \bar{p}(wx + (1-w)) \tag{7}$$

with variance

$$\mathrm{Var}[p(x)] = \mathrm{Var}[\bar{p}(wx - (1-w))] = (\bar{p}w\sigma)^2. \tag{8}$$

For a given value of σ, the systematic factor loading is uniquely defined. As noted by Gordy (2000a) and CreditRisk+ (1997) the value of σ rather determines the shape than the scale of the density function of losses, the latter being determined by the product $w\sigma$. Consequently, the tail of the PDF might be sensible to the chosen value of σ, leading to significantly diverging values for the VaR.

2.2. The computation of the number of defaults

In order to determine the number of defaults within each rating class, we assume conditional independence across defaults. Consequently, the number of defaults within a rating class follows a binomial distribution with parameters $p(x)$, the conditional PD, and y, the number of exposures in the class. For each value of $p(x)$, a random draw is made from the relevant binomial distribution.

Consequently, one needs $R(K-1)$ draws to compute the vectors of default numbers, where R is the number of replications and K the number of risk classes. The number of draws is independent of the size of the portfolio (we made 200.000 drawings in our application). Therefore, the present methodology could be relevant for large SME loans portfolios.

We explicitly treat this step in the simulation. This treatment differs from the CreditRisk+ model one's. The Poisson approximation in this model has the main advantage to provide an analytical form for the loss density function. This approximation fits the data well when PD is small and loans are of large amounts. It might be appropriate for corporate bonds portfolios, but not for small commercial loans portfolios.

So, in order to compute the number of defaults, we have to determine the weight w within each risk class. Because of conditional independence, the probability that two borrowers jointly default is

$$\begin{aligned}&\Pr[Z_i < \Phi^{-1}(\bar{p})\&Z_j < \Phi^{-1}(\bar{p})|x \\ &\quad = \Pr[Z_i < \Phi^{-1}(\bar{p})|x]\Pr[Z_j < \Phi^{-1}(\bar{p})|x] = p(x)^2\end{aligned} \tag{9}$$

with variance

$$\begin{aligned}\mathrm{Var}[p(x)] &= E\lfloor p(x)^2 \rfloor - E[p(x)]^2 \\ &= E[\Pr[Z_i < \Phi^{-1}(\bar{p})\&Z_i < \Phi^{-1}(\bar{p})|x]] - E[p(x)]^2.\end{aligned} \tag{10}$$

Knowing that the latent variables are standard normal variables, with correlation equal to $w^2/(1+w^2)$, the (non-conditional) expected value $E[\Pr[Z_i < \Phi^{-1}(\bar{p})\&Z_i < \Phi^{-1}(\bar{p})|x]]$ is given by the bivariate normal distribution. So, the variance of the conditional PD is

$$\mathrm{Var}[p(x)] = \mathrm{Bivnor}\left(\Phi^{-1}(\bar{p}), \Phi^{-1}(\bar{p}), \frac{w^2}{1+w^2}\right) - \bar{p}^2. \tag{11}$$

Knowing the values of the stationary PD and the variance of the conditional PD, the weight w of the systematic factor is derived as solution of the nonlinear equation (11).

2.3. The allocation of exposures to defaults and the building of the PDF for credit losses: A parametric approach

The object of the last step in the modeling process is to affect exposures to default events or, in other words, to determine which borrowers will effectively default. The most direct solution to this problem is to make random drawings without replacement in each rating class, the number of draws being equal to the number of defaults. This method presents the main advantage of avoiding any assumption concerning the distribution of defaults within a risk class. It is consistent with the implicit assumption of most credit risk models that default events are entirely determined by rating transitions. However, a major drawback of this method is the large number of draws it induces. [4] An alternative method, also used in the CreditRisk+ model, is to determine credit losses by segmenting each rating category in size classes. The total number of default events is then distributed among the size classes on a prorate basis. [5]

Here, we propose another solution, which is suitable for a portfolio containing a large number of small loans. We assume that the distribution of the individual exposures follows a beta distribution. We know the number of defaults in each risk class. So, we make as many draws from the beta distribution as there are defaults in that risk class. [6] The reason for this choice is that the distribution of credit exposures in a loans portfolio is asymmetric. [7] Moreover, the chosen distribution should be bounded in order to compute consistent credit losses.

3. The value at risk of French SME

3.1. The data

The data base we used in our empirical study provided by Coface-SCRL, a large French credit insurance company. It contains two type of information. The first one is a record of 1.364.702 rating grade transitions during the September 1995 to July 1999 period. [8] These transitions concern 224.000 SME. The second information is the bank debt of the same 224.000 SME coming from their annual balance sheets. Notice that our sample is very representative (quasi-exhaustive) of the incorporated SME whose turnover lies between 1 and 500 millions of French francs. It only excludes very small SME (non-incor-

[4] The execution speed of the model is not a major drawback in constructing the density function of credit losses. It becomes a problem when trying to evaluate the accuracy of the credit risk model. Evaluation methods (see Lopez and Saidenberg, 2000) ground essentially on re-sampling techniques, meaning that the model is ran numerous times.

[5] The approximation introduced by this averaging process can be considered as negligible. See CreditRisk+ (1997, Section A4.2).

[6] Modeling the individual credit exposure as a random variable following a given parametric distribution is equivalent to make draws with replacement in the borrowers sample. We assume that the distribution of the exposures is not affected by defaults, what could be the case, for example, if we had drawn exposures within the biggest ones. As defaults remain rare events, we assume that the distribution of credit exposures is not substantially affected by removing defaulting borrowers from the sample, what make draws with replacement possible.

[7] The average bank debt amount in our sample is 1.53 millions French francs, while the standard deviation is 5.68 millions. The skewness of the distribution is equal to 9.37 and the kurtosis is 109.9.

[8] The rating grades were reduced to 7: rating grade 1 corresponds to the lowest degree of credit risk and rating grade 6 to the highest credit risk (the seventh class corresponds to default – that is, in the present case, bankruptcy).

porated firms, like small shops...). Because our simulation of credit exposures uses a beta distribution, which requires the exposure values to be bounded, we exclude firms whose bank debt amount was lower than 10.000 French francs or higher than 100 million French francs. Applying these bounds reduces the number of loans to 194 000, amounting to 344 billion French francs in 1998. However, the PDs and correlation are computed on the whole population.

The first informations were used to compute stationary PDs. More precisely, we took the rating grade of each firm at the beginning of each quarter. Consequently, we only retained one transition within a quarter and neglected other transitions within the same quarter. Then, we computed annual PDs starting from the beginning of each quarter. Thus, we got 11 periods of one year and 11 moving "annual" observations of PDs for each firm over the entire three and a half years period. We computed the weighted mean of these 11 PDs and we took these means as values of the stationary PDs. Moreover, the population was divided in nine categories. To build these categories, we started from an initial classification into 27 two-digit industries combined with a classification in six size classes. Then, tests of aggregation of proportions (corresponding to the stationary PDs) allowed us to reduce these 162 portfolios to nine. Combined with the six rating grades classes, we finally retained 54 portfolios in which we computed the model parameters. Table 1 shows the stationary PDs

Table 1
Stationary PDs

Categories(*)	Rating					
	1	2	3	4	5	6
1	0.0008 (0.0006)	0.0017 (0.0006)	0.0075 (0.0036)	0.019 (0.011)	0.043 (0.02)	0.067 (0.023)
2	0.0022 (0.0028)	0.0021 (0.0004)	0.012 (0.0027)	0.035 (0.017)	0.053 (0.028)	0.108 (0.028)
3	0.0008 (0.0005)	0.0012 (0.0006)	0.0047 (0.002)	0.012 (0.0066)	0.024 (0.01)	0.062 (0.019)
4	0.00019 (0.0004)	0.0007 (0.0004)	0.0021 (0.0008)	0.0054 (0.002)	0.015 (0.009)	0.033 (0.016)
5	–	0.0006 (0.0008)	0.0028 (0.002)	0.012 (0.008)	0.015 (0.012)	0.07 (0.027)
6	–	0.0009 (0.0003)	0.0034 (0.0019)	0.012 (0.007)	0.018 (0.009)	0.036 (0.013)
7	–	0.011 (0.006)	0.023 (0.0029)	0.039 (0.015)	0.043 (0.052)	0.117 (0.05)
8	0.0033 (0.005)	0.006 (0.003)	0.032 (0.014)	0.058 (0.043)	0.09 (0.078)	0.08 (0.058)
9	–	0.0013 (0.0013)	0.016 (0.009)	0.053 (0.014)	0.056 (0.064)	0.094 (0.098)

–: no defaults observed over the period.

Table 2
Correlation between latent variables

Categories	Rating					
	1	2	3	4	5	6
1	1×10^{-6}	0.013	0.027	0.051	0.045	0.029
2	1×10^{-6}	1×10^{-6}	0.006	0.042	0.048	0.008
3	0.0001	0.018	0.019	0.038	0.024	0.018
4	1×10^{-6}	0.018	0.005	0.009	0.032	0.031
5	–	0.04	0.022	0.048	0.004	1×10^{-6}
6	–	0.001	0.03	0.042	0.034	0.022
7	–	0.0015	0.048	0.097	0.158	0.001
8	1×10^{-6}	0.004	0.025	0.1025	0.12	1×10^{-6}
9	–	1×10^{-6}	0.028	1×10^{-6}	0.105	0.125

for the 54 sub-portfolios and their standard deviation in parentheses. Table 2 shows the correlation between latent variables for the same 54 sub-portfolios.

Two remarks can be made concerning Table 1. First, the nine categories show strong differences in average default rates over the period. Though the aggregation criterium is statistical, the nine remaining SME categories are quite homogeneous in terms of activities. For example, the two first categories gather the manufacturing firms, the third one the food industry and consumer goods industry, the fourth one the wholesale industry, and so on. Finally, category 9 gathers mainly specific activities which could not be aggregated with the previous ones. With only 1235 firms, this category could be considered as marginal. The second conclusion is that the default rates show a strong volatility, even in the quite good economic conditions of the second half of the 1990s. A likelihood ratio test (Anderson and Goodman, 1957) unambiguously rejects the null hypothesis of stationary PDs.

From Table 2, the correlation between latent variables is relatively low and is generally increasing in the risk of default, with a noticeable exception for the highest risk class in most categories. This striking result might allow two interpretations. First, this would mean that this rating grade is very near to default and that in times of economic downturn, firms directly move to default without transiting through rating grade 6. Second, this result may show that the assumption of a unique and normally distributed systematic risk factor could not represent the dynamics of the default rate well.

3.2. The results

In order to calibrate the models, it is necessary to know w and consequently to compute the variance of $p(x)$. We adopted the non-parametric method proposed by Gordy (2000a), which is suited for a model with a single sys-

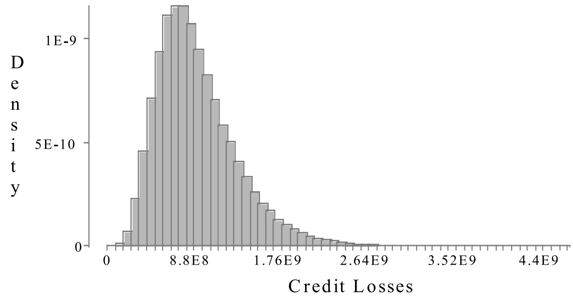

Fig. 1. The PDF for credit losses on French SME bank debt, probit model.

tematic factor. [9] In this approach, the variance of $p(x)$ is determined as follows. Assuming serial independence for the realizations of the systematic factor and conditional independence between defaults, let d_t be the number of defaults in a given year t and n_t the number of borrowers within a given risk class at the beginning of the year t, so that the empirical default frequency for year t is simply defined as $p_t = d_t/n_t$. As explained before, in the computation we got 11 realizations of p_t from our data set. After successive transformations, we get

$$\mathrm{Var}[p(x_t)] = \frac{\mathrm{Var}[p_t] - E[1/n_t]\bar{p}(1-\bar{p})}{1 - E[1/n_t]}. \tag{12}$$

Figs. 1 and 2 present the main outputs of the model: the PDF for credit losses within a planning horizon of one year given, respectively, by the probit and the gamma distributed models (assuming a variance equal to 2 for the systematic risk factor in this model). The density functions were obtained by simulating 200.000 times the aggregated loss and by assuming a recovery rate of 50% (Table 3). Notice that the density functions shown in Figs. 1 and 2 exhibit the expected skewed shape. The hypothesis of a gamma distributed systematic factor leads to a more skewed density function. The Var is defined as the amount of economic capital necessary to cover unexpected credit losses at the chosen confidence level. In other words, the VaR corresponds to the difference between the chosen quantile of the PDF and the mean credit losses (the latter corresponding to the expected credit losses). For the probit model,

[9] Wilson (1988), for example, proposes an alternative parametric method. It consists of developing a multi-factor model, which links the (mostly annual) default probabilities to macroeconomic variables or to reference indexes. Then, this model serves as a basis to forecast the default rates in the economy or in specific industries. However, this methodology imposes to have complete historical series of credit scores, what is necessary to run efficient parametric estimations. Such data are not available for SME and for illiquid loans. This is the main reason why we choose a non-parametric method.

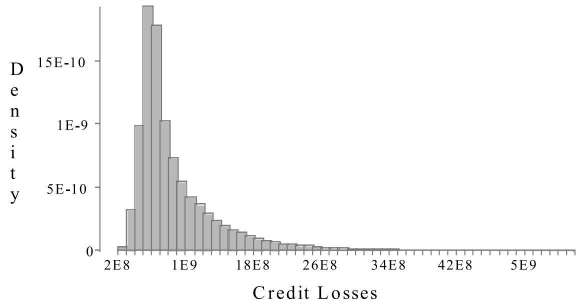

Fig. 2. The PDF for credit losses on French SME bank debt, gamma model.

Table 3
Summary statistics

	Probit model	Gamma model ($\sigma^2 = 2$)
Quantiles		
99%	2.18	3.11
99.5%	2.40	3.59
Max	4.65	8.55
Moments		
Mean	0.896	0.893
Standard deviation	0.407	0.557
Skewness	1.18	2.77
Kurtosis	2.4	11.9

the VaR at the 0.5% level is 1.504 billion French francs. For the gamma model the VaR is higher with 2.697 billion French francs.

4. The measurement of the marginal risk contributions and the allocation of economic capital

The contribution of an exposure to the overall portfolio risk – its marginal risk contribution – is defined by the variation of any aggregated credit losses measure which is induced by the addition of this exposure in the portfolio. In order to compute marginal contributions, one has to determine an allocation rule to split economic capital measured by the VaR. As the Value-at-Risk was computed by using a simulation approach, the distribution function of credit losses can not be characterized analytically. However, it is possible to determine algebraically the marginal contribution of an exposure to the overall variance of credit losses, and a relationship can be established between the

Value-at-Risk and this variance. Indeed, the former can be approximated by a multiple of the standard deviation of the distribution of credit losses:

$$\mu + \xi\sigma = q_\alpha \tag{13}$$

where μ is the mean of the credit losses, q_α is the chosen quantile of the PDF and ξ is the number of standard deviations between q_α and μ. By using this relationship, we can define the marginal contribution to economic capital RCVAR as a multiple of the marginal contribution RC of one exposure to the standard deviation:

$$\mathrm{RCVAR} = \xi\mathrm{RC}. \tag{14}$$

Moreover, the marginal contribution of a given exposure E to the standard deviation can be written as

$$\mathrm{RC} = E\frac{\partial\sigma}{\partial E} \text{ or, equivalently, } \mathrm{RC} = \frac{E}{2\sigma}\frac{\partial\sigma^2}{\partial E}.$$

Recall that, for non-traded loans, the end-of-period value can be represented by a Bernoulli trial, the two states being default and non-default. [10] For correlated Bernoulli trials, the portfolio's variance is given by

$$\sigma^2 = \sum_{i=1}^{N} E_i^2\bar{p}_i(1-\bar{p}_i) + \sum_{\substack{i,j \\ i\neq j}}^{N} E_iE_j\sqrt{\bar{p}_i(1-\bar{p}_i)}\sqrt{\bar{p}_j(1-\bar{p}_j)}\rho(i;j) \tag{15}$$

where $\rho(i;j)$ is the correlation between exposures i and j [11] and $\bar{p}_i$ and $\bar{p}_j$ are the stationary PDs of these exposures i and j. From Eq. (16), we get

$$\mathrm{RC}_i = \frac{1}{\sigma}\left\{E_i^2\bar{p}_i(1-\bar{p}_i) + \sum_{j\neq i}^{N} E_iE_j\sqrt{\bar{p}_i(1-\bar{p}_i)}\sqrt{\bar{p}_j(1-\bar{p}_j)}\rho(i;j)\right\}. \tag{16}$$

The marginal contribution depends on the characteristics of the exposure – its amount E and its stationary PD $\bar{p}_i$ – and on the characteristics of the entire portfolio – the amounts of all other exposures, their PDs and their correlations. [12] The binomial default correlation $\rho(i;j)$ for borrowers i and j is given by

[10] In this calculation, we assume that the recovery rate is equal to zero. The calculation is easily extended to accommodate a positive recovery rate.

[11] This formulation of the portfolio's variance assumes that correlation between default events are independent of the exposures, reflecting the fact that default events are assumed to be independent of the level of the exposure.

[12] Moreover, the marginal contributions RC to the standard deviation sum up to the standard deviation by a general property of homogeneous polynomials.

$$\rho(i;j) = \frac{\alpha - \bar{p}_i\bar{p}_j}{\sqrt{\bar{p}_i(1-\bar{p}_i)}\sqrt{\bar{p}_j(1-\bar{p}_j)}} \tag{17}$$

where α is the joint PD. Using the previous notations, the joint probability α is given by

$$\alpha = \text{Bivnor}\left(\Phi^{-1}(\bar{p}_i), \Phi^{-1}(\bar{p}_j), \frac{w_i w_j}{\sqrt{(1+w_i^2)(1+w_j^2)}}\right). \tag{18}$$

In this way, we were able to compute the marginal contribution RCVAR for each exposure.

5. Comparison between the capital requirements derived from the internal model and the capital requirements derived from the IRB capital regulation

The New Capital Accord proposes to compute the regulatory capital requirements by using an IRB approach. In this framework, the risk weights for retail exposures R and corporate exposures C are given by the following formulae:

$$\text{BRW}_R(\bar{p}) = 9.765\Phi(1.043\Phi^{-1}(\bar{p}) + 0.766)(1 + 0.047(1-\bar{p})/\bar{p}^{0.44}), \tag{19}$$

$$\begin{aligned}\text{BRW}_C(\bar{p}) = {} & 9.765\Phi(1.118\Phi^{-1}(\bar{p}) + 1.288)(1 + 0.047(1-\bar{p})/\bar{p}^{0.44}) \\ & \times (1 + b(\bar{p})(M-3)),\end{aligned} \tag{20}$$

$$\text{with } b(\bar{p}) = \frac{0.0235(1-\bar{p})}{\bar{p}^{0.44} + 0.047(1-\bar{p})},$$

where M is the effective remaining maturity and $b(\cdot)$ is a maturity adjustment (the assumed benchmark maturity is three years). Moreover, these risk weights must be adjusted to account for the recovery rate in case of default. The IRB framework assumes a benchmark recovery rate of 50% in its foundation approach. Having no specific information on recovery rates in our data base, we will assume a fixed recovery rate of 50% in what follows. The parameters of the IRB framework are calibrated in such a way that one obtains a 100% risk weight for a PD of 0.7% (and a maturity of three years for a corporate exposure). This is also equivalent to assume a constant value of 0.45 (resp. 0.28) for the systematic risk factor weight w, i.e. a correlation between latent variables of approximately 20% (8%) for loans termed as corporate (resp. retail). These IRB assumed values contrast with the average correlation in our SME sample which is about 2%. In order to foster the comparison between our

Table 4
Capital requirements (in billions French Francs)

	Corporate approach $\rho = 0.20$	Retail approach $\rho = 0.08$	Sample correlation
IRB	12.54	9.14	–
Gamma model $\sigma^2 = 2$	8.91	6.63	3.59
Probit model	8.14	4.10	2.40

Note: Capital requirements according to the current Basel ratio are 27.5 billions.

models and the IRB weighting scheme, we computed the 99.5% VaR accounting for these differences in correlation. Concerning the probit model, the estimated correlation (Eq. (11)) was simply replaced by the IRB assumed values. Concerning the gamma distributed model, an estimated conditional variance was calculated by using Eq. (11) and the assumed correlation values. This variance was then introduced in Eq. (12) in order to compute a new value for the weight of the gamma distributed systematic factor. The results are gathered in Table 4. Though we used similar parameter estimates in the calculations, the different models still led to strongly different capital requirements. This stems from the fact that the IRB framework introduces a scaling factor that leads to a much more stringent bankruptcy criteria than the one implicitly assumed in the 99.5% rate of coverage of expected and unexpected losses. This scaling factor can also be interpreted as a correction factor which accounts for the fact that the planning horizon of one year, which is the chosen horizon of most credit risk models, might be too short. Indeed, in the case of a severe recession or a systemic crisis, a one year planning horizon does not take of the persistence of large loan losses over several periods into account.

Fig. 3 depicts the distribution of capital requirements given by the IRB retail approach, the gamma model and the probit model with a 99.5% loss coverage. The plain horizontal line represents the current capital requirement of 8% all loans being 100% weighted. The three curves show the same exponential increase with the most risky exposures. Unsurprisingly, the IRB capital requirements lie much higher than those computed by the internal model. However, if we assume that all loans in our SME portfolio could be considered as retail loans, the New Basel Accord would lead to a sharp decrease in regulatory capital, compared to the actual capital ratio. [13]

However, equally risky capital exposures could be of different amounts. Fig. 3 does not take such differences into account. Consequently, it does not represent the amount of capital consumption corresponding to the most risky segments of the portfolio. In order to compare the capital requirements given by the regulatory scheme and the credit risk model more accurately, we present

[13] A similar result would be observed assuming all loans are corporate loans.

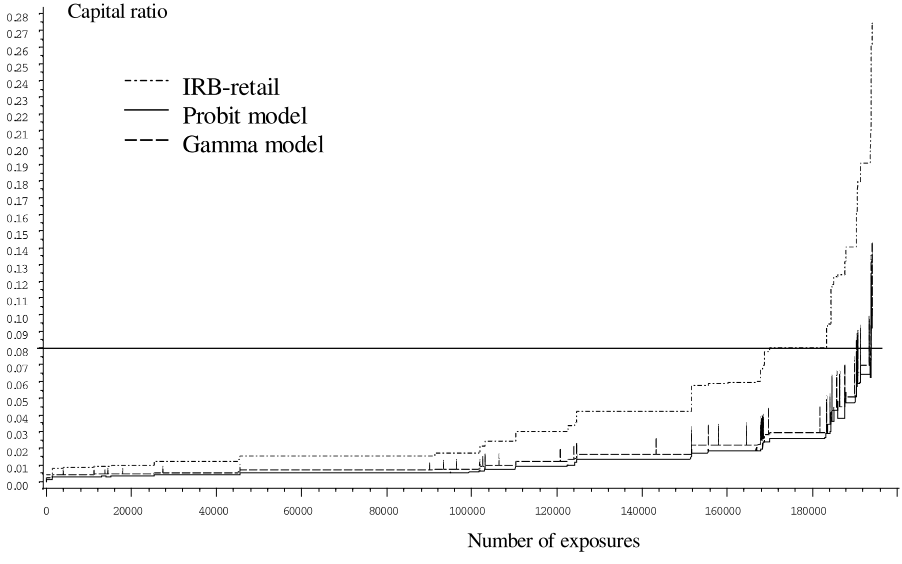

Fig. 3. Distribution of capital requirements rates.

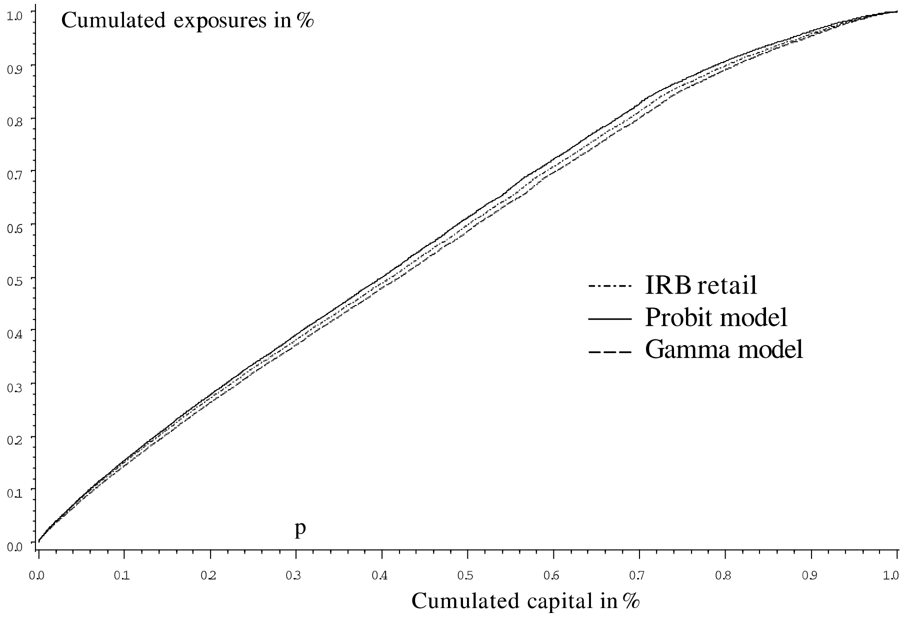

Fig. 4. Capital requirements concentration curves.

capital concentration curves in Fig. 4. One can see, for instance, that considering the probit model, the 10% most risky exposures consume about 20% of the total computed bank capital. Alternatively, the 20% less risky exposures only consume around 10% of the required bank capital.

Notice that the three measures of the capital requirements lead to a very similar distribution of capital across exposures, despite the diverging levels of capital requirements. Moreover, the higher correlation in the IRB approach does not seem to affect the distribution of capital requirements, but their level. The observed differences stem from the differences in the treatment of the correlations. On one hand, the probit model uses specific intra- and inter-group correlation, while, on the other hand, the gamma model and the IRB model use average correlations. However, the differences appear to be of a reduced order of magnitude, as shown by Fig. 4. The probit model leads to allocate 22% of total regulatory capital to 10% most risky exposures, against respectively 19% and 18% for the IRB and gamma models. Consequently, the choice of a particular allocation scheme for regulatory capital implies only moderate distortions among borrowers.

6. The pricing of SME commercial loans

The marginal risk contribution of an exposure to the portfolio's Value-at-Risk can be used to measure risk-adjusted loan prices. The issue of the loan pricing can be treated as a portfolio allocation problem. We assume that a lender maximizes the expected return of his portfolio P under the constraint that the economic capital requirement K_{p} must be equal to a (exogenous) given amount V. Formally, the lender's objective function is

$$\underset{E_i}{\mathrm{Max}}[r_{\mathrm{p}}] = \sum_i E[r_i]E_i,$$
$$K_{\mathrm{p}} = V,$$

where $E[r_{\mathrm{p}}]$ is the loans portfolio's expected return, $E[r_i]$ is the loan i expected return, and E_i is the value of the exposure i. Multiplying the First Order Condition of this program by E_i yields

$$\frac{E[r_i]E_i}{\partial K/\partial E_i} = \lambda E_i. \tag{21}$$

The left-hand side of Eq. (21) represents the expected return of the equity capital allocated to loan i. In other words, it is the RoE required by the lender. Assuming a one year maturity, it is possible to determine the minimal loan price consistent with the lender target RoE. Indeed, Eq. (21) can be rewritten as

$$\lambda E_i = 1 + \mathrm{RoE}$$
$$= \frac{(1+r_l)(1-\bar{p})L + \bar{p}\tau(1+r_l)L - [L - \mathrm{RCVAR} - \bar{p}(1-\tau)L(1+r_l)](1+r_f)}{\bar{p}L(1+r_l)(1-\tau) + \mathrm{RCVAR}}$$
$$\Longleftrightarrow r_l = \frac{\mathrm{RCVAR}(\mathrm{RoE} - r_f) + L(1+r_f)}{[(1-\bar{p}) + \bar{p}\tau + (r_f - \mathrm{RoE})\bar{p}(1-\tau)]L} - 1 \qquad (22)$$

where r_l is the loan price, L is the loan amount, r_f is the risk-free rate and τ a fixed recovery rate. r_l is simply the risk adjusted prices a lender could charge in order to reach (on average) his target RoE.

Table 5 (Panel A) shows the average price of bank debt in the 54 sub-portfolios of our sample. We assumed a 6% risk-free rate and a 15% return on equity. Taxes as well as operating costs were neglected. The loans prices of Table 4 define a risk structure of credit spreads. As expected, this structure is increasing, except in two cases (grade 2 in category 2 and grade 5 in category 8). In these two cases, the stationary PD is non-monotonic in the risk grades. This leads to decreasing loan prices. Moreover, such a non-monotonic property could reflect the differences in correlations between sub-portfolios. But, as shown in table, the correlation between latent variables is generally increasing

Table 5
Average interest rates on SME commercial loans, probit model (Panel A) and average IRB interest rates on SME commercial loans, retail approach (Panel B)

Categories[a]	Rating					
	1	2	3	4	5	6
Panel A						
1	0.0608	0.0614	0.0653	0.0727	0.088	0.103
2	0.0618	0.0616	0.068	0.0832	0.0945	0.13
3	0.0608	0.061	0.0634	0.0684	0.0758	0.0997
4	0.0602	0.0607	0.0616	0.0639	0.0698	0.0816
5	0.06	0.0606	0.0621	0.0685	0.07	0.105
6	0.06	0.0608	0.0625	0.0682	0.072	0.083
7	0.06	0.0678	0.0759	0.0862	0.0892	0.137
8	0.0626	0.0642	0.0814	0.0993	0.121	0.112
9	0.06	0.0611	0.0706	0.0938	0.0976	0.124
Panel B						
1	0.0613	0.0624	0.068	0.078	0.097	0.116
2	0.0629	0.0628	0.072	0.091	0.105	0.149
3	0.0613	0.0617	0.0654	0.0723	0.0822	0.112
4	0.0605	0.0612	0.0628	0.0661	0.0745	0.0897
5	0.06	0.0611	0.0635	0.0724	0.0748	0.119
6	0.06	0.0614	0.0641	0.0722	0.0773	0.0916
7	0.06	0.0711	0.0817	0.0944	0.0979	0.156
8	0.064	0.0665	0.0887	0.109	0.135	0.127
9	0.06	0.062	0.075	0.105	0.108	0.138

[a] See Section 3 for the definition of the groups.

with the risk of default. In the cases where the correlation is decreasing, the rise in PDs still leads to a higher capital consumption and to higher risk adjusted prices. Moreover, the high level of loans prices in the highest risk classes should rather be interpreted as a signal that the lender would reject the loan application. A particular case arises when the PD is zero, that is when no defaults were observed in that sub-portfolio over the planning horizon. In that case, no economic capital is needed and, following Eq. (22), the loan price is the risk-free rate.

As illustrated by Table 5 (Panel B), the IRB retail approach leads to higher loan prices than the probit model, with a difference of more than 100 basis points in higher risk classes. However, results also show that the differences are very small for the lower risk classes. Moreover, the IRB corporate approach (not shown here) implies an additional percentage point in the cost of funds, again for the riskier segments of the loans portfolio. Consequently, in a risk adjusted loan pricing scheme, the SME loan price will be sensitive to the classification as retail or corporate exposures.

By deriving Eq. (22) with respect to L, it can be shown that loan price increases with the size of the loan. However, our results show that the loan prices observed differences induced by the exposure's size are relatively low: the average gap between the highest and the lowest loan prices in each portfolio is only 25 basis points. As shown by Gordy (2000b), if the size of the portfolio is large, the capital requirements become portfolio invariant, i.e. the relative capital charge becomes independent of the size of the exposure for given values of PDs and correlation. This so-called "granularity" condition can be considered as satisfied in our SME population.

7. Conclusions

In this paper, we proposed an internal credit risk model for SME loans. This model allowed us to compute the Value-at-Risk of any large portfolio of small commercial loans and to derive the allocation of capital and loans pricing schemes for this kind of loans. The methodology was applied to a very large sample of French SME data base. Given the size of the data base, we were able to measure the global risk of all the French SME sector, as if one single bank would own all the small and medium businesses loans in its portfolio.

Some conclusions can be drawn from our applications. First, the capital requirements derived from an internal model are significantly lower than those derived by the standard capital ratio and the new IRB approach as well. These differences between the capital requirements could be partially explained as a consequence of the low values of the default correlations in our SME sample over the 1995–1999 period we considered (during this period, the state of the economy was rather good). Over a longer time period, one likely would have observed larger values of the default volatilities, what could have increased the correlation values. These differences could also be the consequence of the choice of a short – one year – planning horizon. Over a longer horizon, losses would accumulate, which would make it necessary to increase the capital requirements level. These differences could also be the consequence of the diversification effects stemming from the large size of the sample. In smaller loan portfolios, the diversification would be lower, which would induce higher capital requirements. Moreover, our results demonstrate the interest of internal credit risk models explicitly taking into account the correlation among the exposures, even in the case of retail portfolios. Secondly, the results verify that one of the main advantages of an internal credit risk model is to lead to a better allocation of capital and to better loan pricing.

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